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Additive and Multiplicative Risk Premiums with
Multiple Sources of Risk

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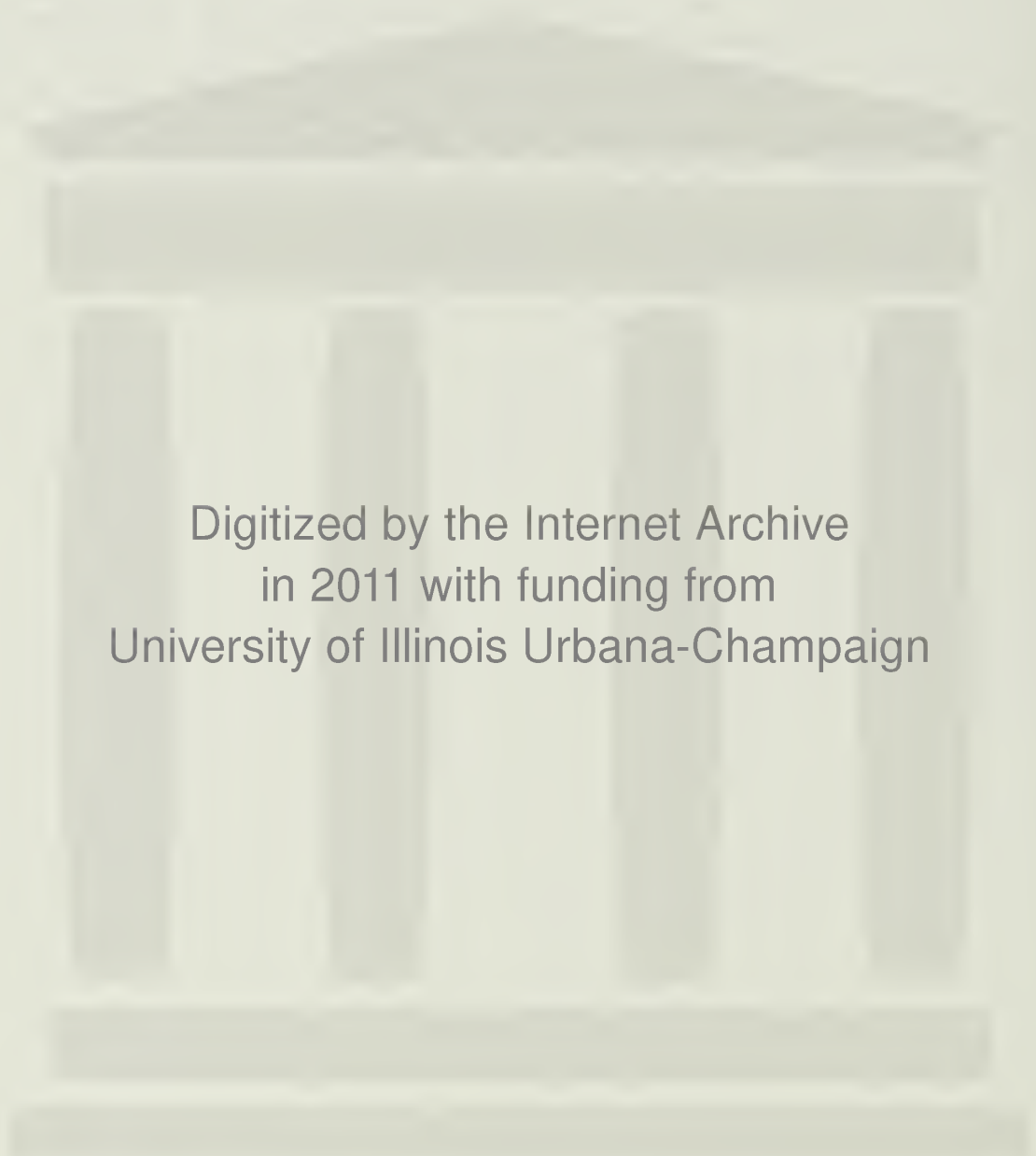with Multiple Sources of Risk

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Abstract

This paper considers the risk premium for one risk when other background risk is present. In a mean-variance setting, we examine the conditions under which the risk premium will be negative. These conditions consider the variance of the considered risk in relation to the covariance of the considered risk with the background risk. We consider the cases of both multiplicative and additive risks. A breakdown of the total risk premium into several components is established by removing each source of risk consecutively. The case of mutliplicative risks is shown to be quite different from the additive-risk case in that the individual's total willingness-to-pay for the removal of all risk on a sequential basis does not necessarily equal the willingness-to-pay for removal of all risk simultaneously.

I.   Introduction

A recent set of papers (Kihlstrom, Romer and Williams (1981), Nachman (1982), Pratt (1982) and Ross (1981)) has examined measures of risk aversion when initial wealth is randomly distributed. These papers deal with aversion to one risk when other risks are present, and their results have already had some far-reaching consequences for insurance markets as evidenced by recent papers by Doherty and Schlesinger (1983) and Turnbull (1983). However, these papers have focused mainly on the local properties of risk aversion. In this paper, we concern ourselves with risk aversion for finite discrete risks. In particular, we examine the nature of risk premiums in a model with multiple sources of risk. We show that willingness-to-pay for the sequential removal of risks is identical to willingness-to-pay for the simultaneous removal of all risk when the multiple risks are additive or when they are multiplicative but with a zero covariance. We also examine conditions under which the risk premium will be negative.

Pratt (1964) demonstrated an equivalence between the sign and ordering of the Arrow-Pratt measure of local risk aversion and the sign and ordering of the risk premium payable on any given risky transaction. A risk averter would pay a positive risk premium and a more-risk-averse person would pay a higher risk premium on the same risky transaction. However, this ordering equivalence need not be preserved when background risk is present, even if the transaction risk and background risk are independent, as was demonstrated by Kihlstrom, Romer and Williams

(1981-hereafter KRW). Furthermore, once the independence assumption is dropped, this sign equivalence need not be preserved either. KRW showed that risk premium for a risk averte: could be negative if there is a negative correlation between sources of risk. However, a positive risk premium is also possible, even with perfect negative correlation. This is not surprising once one considers the stochastic relationship betwee: sources of risk in terms of the total-wealth prospect.

In this paper we derive a method for "adding up" risk premiums (or equivalently "breaking them down") based on the stochastic nature of wealth. This derivation is developed via considering the sequential elimination of sources of risk in two alternative frameworks. In the first framework (section II) additive risks are considered, whereas multiplicative risks are assumed in the second framework (section IV). For each of these models a simple example is provided which illustrates the "adding up" formula and allows for a graphical representation of both positive and negative risk premiums (sections III and V). The main results are summarized in section VI.

II. Risk premiums with additive risks

We define the individual's total wealth prospect as $\tilde{Y}=\tilde{w}+\tilde{x}$ where $(\tilde{w},\tilde{x})$ is a random pair taking values in E^2. The case where w is nonrandom represents the standard single risk model. The individual is assumed to be strictly risk averse with preferences

expressed by the twice differentiable von Neumann-Morgenstern utility function, $U(Y)$, where $U'(Y)>0$ and $U''(Y)<0$. We consider the following three distinct risk premiums:

    (i)   The total risk premium for removing all risk from the individual's wealth portfolio, π_y, which is defined by

(1) $EU(\tilde{w}+\tilde{x}) = U(E\tilde{w}+E\tilde{x}-\pi_y)$.

The symbol E, above, denotes mathematical expectation. For a risk averter, π_y is always positive.

    (ii)   The risk premium of $\tilde{x}$ in the presence of $\tilde{w}$, π_x, defined by

(2) $EU(\tilde{w}+\tilde{x}) = EU(\tilde{w}+E\tilde{x}-\pi_x)$.

Thus, π_x is the premium one is willing to pay to have $\tilde{x}$ replaced by its expectation while retaining the riskiness of $\tilde{w}$. The amount $E\tilde{x}-\pi_x$ is the certainty equivalent for $\tilde{x}$ in the presence of $\tilde{w}$, which we denote by C_x. The individual is willing to replace $\tilde{x}$ by any certain amount exceeding C_x, while maintaining the randomness in $\tilde{w}$.

    (iii) The conditional risk premium for removing the risk in $\tilde{w}$ _after_ having paid to remove the riskiness in $\tilde{x}$. We denote this risk premium $\pi_{w|C_x}$, defined by

(3) $EU(\tilde{w}+C_x) = U(E\tilde{w}+C_x-\pi_{w|C_x})$.

Thus, $\pi_{w|C_x}$ represents a willingness-to-pay to remove the riskiness in $\tilde{w}$, given that $\tilde{x}$ has already been replaced by C_x.

We note that both π_y and $\pi_{w|C_x}$ are ordinary Arrow-Pratt risk premiums. The former treats $\tilde{w}+\tilde{x}$ as a single random variable (i.e., a single source of risk) while the latter has only one source of risk to consider, namely $\tilde{w}$. However, π_x is not the usual Arrow-Pratt type of risk premium; rather, it is the risk premium used by KRW and it removes only part of the total risk.

From equations (1), (2) and (3), the following result is immediate:

Proposition 1A: $\pi_y = \pi_x + \pi_{w|C_x}$.

This result is fairly obvious, saying that the maximum amount of income an individual would pay to remove all riskiness in $\tilde{w}+\tilde{x}$ must equal the sum of the amount he would pay to remove the riskiness in $\tilde{x}$ alone and the amount he would pay to remove the remaining risk in $\tilde{w}$ after having paid π_x to remove the risk in $\tilde{x}$. However, as we shall see in section IV, this nice additivity propoerty does not always hold when risks are multiplicative.[1]

Another result that follows straight from the definitions and from Jensen's inequality is

Proposition 2A: If w and x are independently distributed, $\pi_x > 0$.

When $\tilde{w}$ and $\tilde{x}$ are not independently distributed, it is easy to see that π_x might be negative when the correlation is negative. For example, if $\tilde{x} = -\tilde{w}$, it is easy to show that $\pi_y = 0$, $\pi_x < 0$ and $\pi_{w|C_x} = -\pi_x$.

We can gain some insight into this situation if we first consider a case where the riskiness of $\tilde{w}$ and of $\tilde{x}$ is very small. In a manner similar to Pratt (1964) and Ross (1981), we can take Taylor expansions of each side of (1). This provides an approximation for π_x in terms of the covariance of $\tilde{x}$ and $\tilde{w}$. Expanding (1) around $E\tilde{w}+E\tilde{x}$, we obtain the following approximation: [2]

$$(4) \quad (1/2) \operatorname{Var}(\tilde{w}+\tilde{x})U'' = -\pi_x U' + (1/2) \operatorname{Var}(\tilde{w})U''$$

where U' and U'' are evaluated at $E\tilde{w}+E\tilde{x}$. Noting that $\operatorname{Var}(\tilde{w}+\tilde{x}) = \operatorname{Var}(\tilde{w}) + \operatorname{Var}(\tilde{x}) + 2\operatorname{Cov}(\tilde{w},\tilde{x})$ and dividing by $(-U')$ yields

$$(5) \quad \pi_x = \left[(1/2) \operatorname{Var}(\tilde{x}) + \operatorname{Cov}(\tilde{w},\tilde{x}) \right] R$$

where $R = -U''/U'$ is the local measure of absolute risk aversion evaluated at $E\tilde{w}+E\tilde{x}$.

Although (5) is an approximation, it indicates that a necessary, but not sufficient, condition for π_x to be negative is that $\mathrm{Cov}(\tilde{w},\tilde{x})$ be negative. Indeed, we observe that $\pi_x < 0$ if and only if $\mathrm{Cov}(\tilde{w},\tilde{x}) < -(1/2)\,\mathrm{Var}(\tilde{x})$.

III. Additive risks: an illustration

In order to gain some geometric intuition, consider an example where $(\tilde{w},\tilde{x}) = (w_o, x_1)$ or (w_1, x_o) with probability 1/2 each, and where $w_o < w_1$ and $x_o < x_1$. Thus, $\tilde{w}$ and $\tilde{x}$ have a correlation of -1. $\pi_x > 0$ if and only if $EU(\tilde{w} + E\tilde{x}) > EU(\tilde{w} + \tilde{x})$. In our case this reduces to

$$U(w_1 + E\tilde{x}) - U(w_1 + x_o) > U(w_o + x_1) - U(w_o + E\tilde{x})$$

or

$$U(w_1 + x_o + (E\tilde{x} - x_o)) - U(w_1 + x_o) > U(w_o + E\tilde{x} + (x_1 - E\tilde{x})) - U(w_o + E\tilde{x})$$

Noting that $E\tilde{x} - x_o = x_1 - E\tilde{x}$, one may divide both sides by the same amount and obtain, under the small risk assumption

$$U'(w_1 + x_o) > U'(w_o + Ex).$$

and finally, $\quad 2(w_1 - w_o) < x_1 - x_o.$

In other words, even with perfect negative correlation, Π_x will be positive if the spread in $\tilde{w}$ is relatively small. More generally, it follows from (5) that Π_x will be positive, assuming small risks with a perfect negative correlation, whenever $\text{Var}(\tilde{w}) < (1/4)\text{Var}(\tilde{x})$. The above example is illustrated in figures 1-3.

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In figure 1, $\tilde{w}+E\tilde{x}$ is preferred to $\tilde{w}+\tilde{x}$. Consequently, we can subtract π_x from each of the $\tilde{w}+E\tilde{x}$ values to obtain a new prospect, (w_o+C_x) or (w_1+C_x) each with probability 1/2, which is indifferent to $\tilde{w}+\tilde{x}$. Once π_x is determined, it is easy to illustrate the "adding-up" formula given by Proposition 1A. To avoid clutter, this is shown in figure 2, which reproduces some essentials from figure 1. Both figures 1 and 2 illustrate the case where $\pi_x = 0$. An example where $\pi_x < 0$ is illustrated in figure 3.

IV. Risk premiums with multiplicative risks

It has been shown by Turnbull (1983) that the stronger risk aversion measure proposed by Ross (1981) is not strong enough to provide meaningful results when risks are multiplicative rather than additive. Therefore, it seems worthwhile to investigate whether the results obtained in section II under the additivity assumption are preserved under multiplicative risks.

We redefine the individual's final wealth propsect as $\tilde{Y} = w + \tilde{z}\tilde{x}$, where

. w is his nonrandom initial wealth;

. x is a speculative risk faced by the individual, for example the random market value of a property at risk;

. $z \epsilon [0,1]$ is a pure risk, for example, the extent of a damage threatening the individual's property.

As before, we may define three distinct risk premiums:

i)   The total risk premium π_y which the individual would b willing to pay for removing both risks inherent in $\tilde{Y}$, defined by

(6) $EU(w + \tilde{z}\tilde{x}) = U(w + E(\tilde{z}\tilde{x}) - \pi_y)$.

By Jensen's inequality, $\pi_y > 0$ for all strictly concave U.

ii)   The risk premium for $\tilde{x}$ in the presence of $\tilde{z}$, π_x, which is the premium that the individual would be willing to pay to obtain the expected value of x instead of the random amount x, while maintaining the randomness of $\tilde{z}$ It is given by

(7) $EU(w+\tilde{z}\tilde{x}) = EU(w+\tilde{z}E\tilde{x}-\pi_x)$

$= EU(C_x+\tilde{z}E\tilde{x})$,

where $C_x=w-\pi_x$. In this case C_x is nonrandom again, but it is not the certainty equivalent of $\tilde{x}$, as in section II.

iii) The condition risk premium $\pi_{z|C_x}$, which is the amount that the individual would be willing to pay to eliminate the pure risk $\tilde{z}$ and replace it by $E\tilde{z}$, after having replaced the risk $\tilde{x}$ by its expected value. It is given by

(8) $EU(C_x+\tilde{z}E\tilde{x}) = U(C_x+E\tilde{z}E\tilde{x} - \pi_{z|C_x})$.

By Jensen's inequality $\pi_{z|C_x} > 0$ for all strictly concave U.

From these definitions, we obtain the following result:

Proposition 1M: $\pi_y = \pi_x + \pi_{z|C_x} + Cov(\tilde{z},\tilde{x})$.

Thus, when risks are multiplicative, Proposition 1A does not hold in general: the global risk premium differs from the sum of the two partial risk premiums. The additivity property of risk premiums holds only under independence of risks, or under the somewhat less restrictive assumption of zero correlation. If, for example, $Cov(\tilde{z},\tilde{x}) < 0$, Proposition 1M indicates that the individual would pay less to remove both risks simultaneously than to remove them one after the other. Lest we think arbitrage

opportunities would prevent this from occurring, we must point

out that we do not end up at the same replacement for $\tilde{z}\tilde{x}$ in the

two cases. When the risk is removed all at once, we replace $\tilde{z}\tilde{x}$

with $E(\tilde{z}\tilde{x})$. On the other hand, replacing the random variables

sequentially leaves us with $(E\tilde{z})(E\tilde{x})$. Thus, the amount we are

willing to pay for these replacements is not expected to be the

same. It is trivial to show that the total amount we would pay t

remove the risks sequentially does not depend upon the order of

their removal.

As before, we are interested in the conditions under which

$\pi_x < 0$.

Proposition 2M: If $\tilde{x}$ and $\tilde{z}$ are independently distributed,

$\pi_x > 0$.

Proof: Straightforward from Jensen's inequality   Q.E.D

If $\tilde{x}$ and $\tilde{z}$ are not independently distributed, Proposition 2

will clearly not hold in all cases. An example is provided in th

illustration below. We also note that the relationship between π

and π_x is not as simple as it was in section II. In some cases,

the global risk premium π_y may be smaller than the partial risk

premium π_x. From Proposition 1M, we see that this will occur

whenever $\text{Cov}(\tilde{z}, \tilde{x}) < -\pi_{z|C_x}$.

V.   Multiplicative risks: an illustration

Using the model presented in the preceding section, we provide here a two-state illustration. Let $x \epsilon \{x_o, x_1\}$, $x_1 > x_o$, and let $z \epsilon \{0, 1\}$, i.e. full loss or no damage at all. Moreover assume perfect positive correlation, i.e. $x = x_1$ when $z = 1$ and $x = x_o$ when $z = 0$. We thus have:

$$\tilde{Y} = \begin{cases} w + x_1 \\ \\ w \end{cases}$$

From (7), the condition for $\pi_x < 0$ is

$$EU(w + \tilde{z}\tilde{x}) > EU(w + \tilde{z}E\tilde{x}).$$

Using the assumptions above, this reduces to $x_1 > E\tilde{x}$, which is always satisfied. Thus π_x is negative under perfect positive correlation. By a similar argument, it may be shown that π_x is necessarily positive under perfect negative correlation.

These results are easily explained. Under positive correlation, the damage is high when the market value of the property is low. The individual is not prepared to pay much to remove the volatility in $\tilde{x}$ because he would lose on positive deviations without earning something on negative deviations. Hence π_x is low and may even be negative. Inversely, under

negative correlation, the individual is prepared to pay a high π_x because this would remove the negative deviations on $\tilde{x}$, without having to incur a loss on positive deviations.

VI. Concluding remarks

We have presented some results on risk premiums with multiple sources of risks by considering first an additive risk framework and then a multiplicative risk framework. In the first case we find that the risk premium for the removal of all risks equals the sum of the risk premium for removing one risk while retaining the other and the risk premium for removing the remaining risk after having paid the risk premium to remove the first risk. This nice relationship does not hold any longer when risks are multiplicative. The covariance between the risks is then also a component of the total risk premium. In addition, we show that the risk premium for removing one risk while retaining the other may be negative, both in the additive and multiplicative risk frameworks. Negative correlation is a necessary condition for this result to hold when risks are small additive. Inversely, a simple illustration shows that the result is associated with positive correlation under multiplicative risks.

Most previous work concerns itself with extending canonical results concerning an individual being "more risk averse" to the case of multiple sources of risk. For example, Pratt (1982) considers $\pi(\tilde{x}|w)$, defined by $EU(w+\tilde{x})=U(w+E\tilde{x}-\pi(\tilde{x}|w))$ where w is a nonrandom realization of $\tilde{w}$. Pratt then shows that the usual

measure of risk aversion, $-U'(Y)/U''(Y)$ is not sufficient to guarantee that "more risk averse" implies: "π_X is larger"--even when $\tilde{w}$ and $\tilde{x}$ are independent. Pratt goes on to show that additional assumptions concerning $\pi(\tilde{x}|w)$ are sufficient to yield the desired conclusions.

The focus of this paper is not on comparing different individuals exposed to the same risky prospect. We focus on the nature of the risk; in particular, on the relationship between the sources of risk. The Arrow-Pratt measure of local risk aversion works well for what it is designed to do; namely, place a subjective value on the risk inherent in the total-wealth prospect. When considering the removal of only one of many sources of risk, it is not surprising that we do not necessarily preserve the sign and ordering equivalence between the risk premium and the Arrow-Pratt measure of risk aversion.
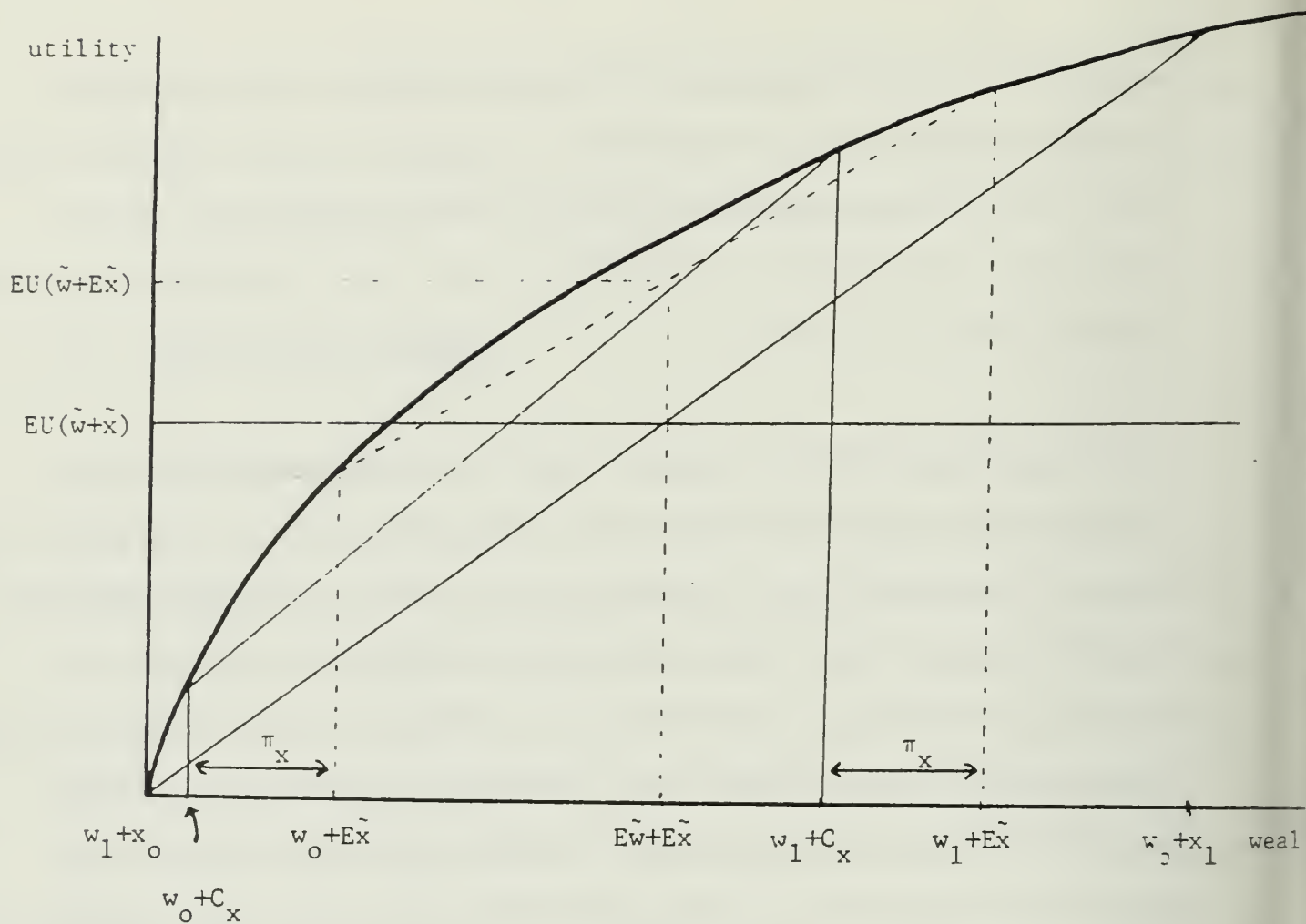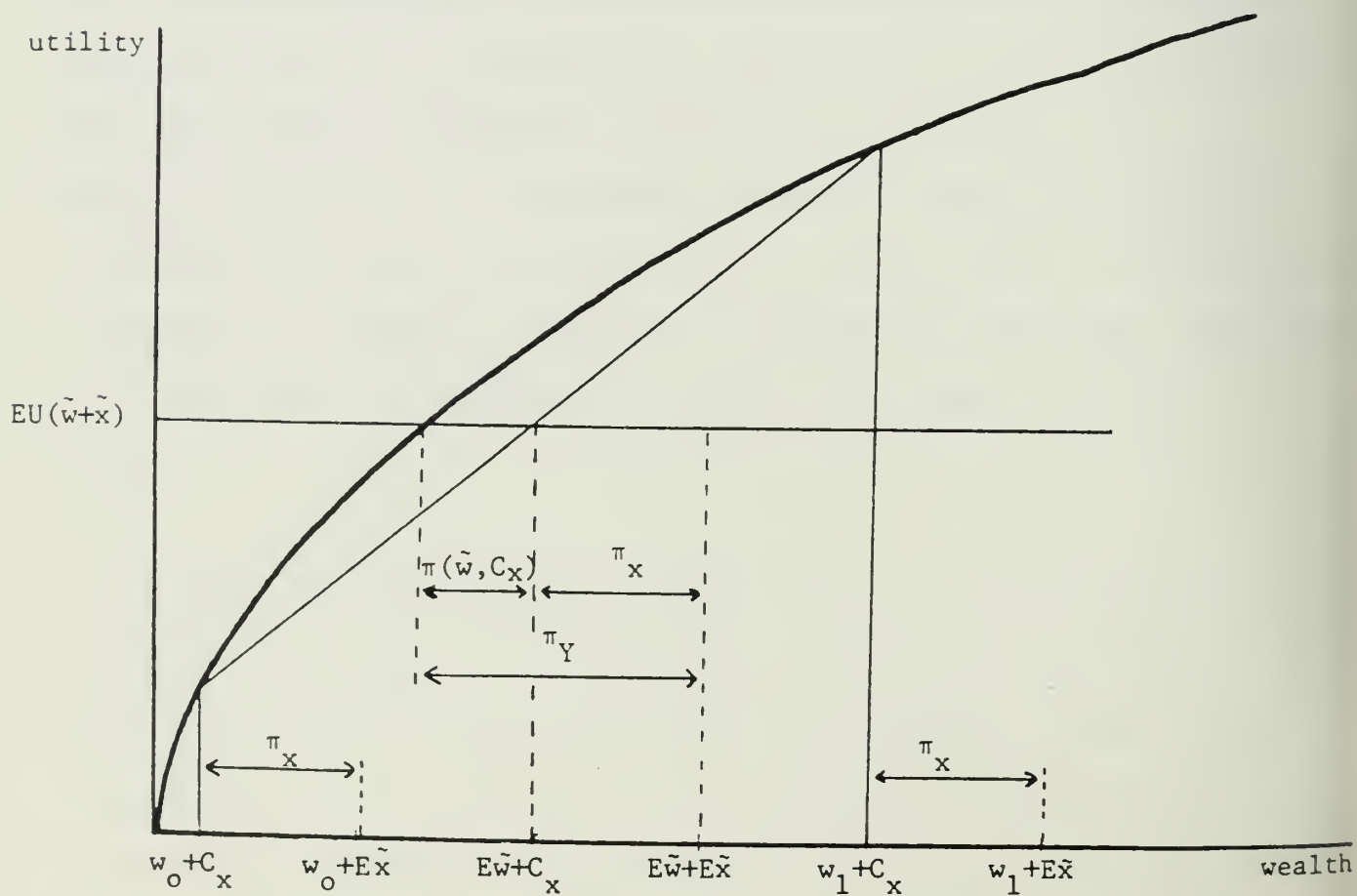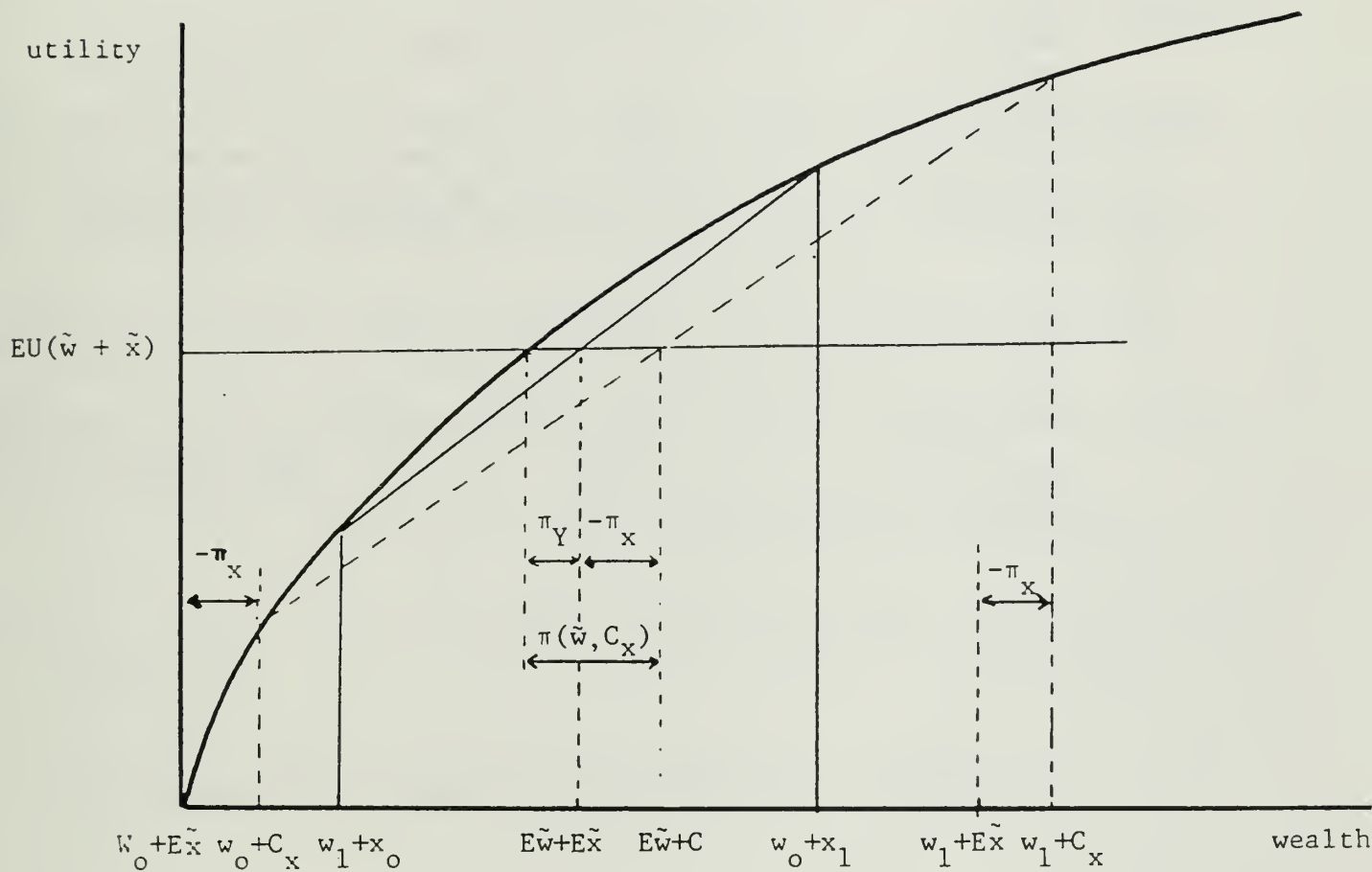
Figure 1

Figure 2

Figure 3

Footnotes

1. Although we only consider two risky sources, the results are extendable to three or more sources of risk in a straight-forward, but tedious, manner. It is also possible to examine the distribution of $\tilde{y}$ compared with the convolution of the marginal distributions of $\tilde{x}$ and $\tilde{w}$ to derive stochastic dominance rules for additive risk premiums. See Doherty and Schlesinger (1986).

2. Note that we have approximated the left-hand side (LHS) of equation (1) with a first-order expansion, while we used a second-order expansion for the right-hand side (RHS). This yields errors of asymptotic orders $O(\pi^2)$ and $o[Var(\tilde{x})]$ respectively. Thus, a higher-order expansion of the LHS would not increase the accuracy of the approximation for small $Var(\tilde{x})$. See Pratt (1964) and Ross (1981) for a further discussion.

References

N. Doherty and H. Schlesinger (1983), "The Optimal Deductible for an Insurance Policy when Initial Wealth is Random," Journal of Business, 56, 555-565.

N. Doherty and H. Schlesinger (1986), "A Note on Risk Premiums with Random Initial Wealth,"Insurance:Mathematics and Economics, 52,183-186.

R. Kihlstrom, D. Romer and S. Williams (1981), "Risk Aversion with Random Initial Wealth," Econometrica, 49, 911-921.

D. Nachman (1982), "Preservation of 'More Risk Averse' Under Expectations," Journal of Economic Theory, 28, 361-368.

J. Pratt (1964), "Risk Aversion in the Small and in the Large," Econometrica, 32, 122-136.

_____(1982), "Aversion to One Risk in the Presence of Others," unpublished working paper, Harvard University.

S. Ross (1981), "Some Stronger Measures of Risk Aversion in the Small and in the Large with Applications," Econometrica, 49, 621-638.

S. Turnbull (1983), "Additional Aspects of Rational Insurance Purchasing," Journal of Business, 56, 217-229.